The Market portfolio (M) is the tangent portfolio.
The Market portfolio (M) is on the EEF.

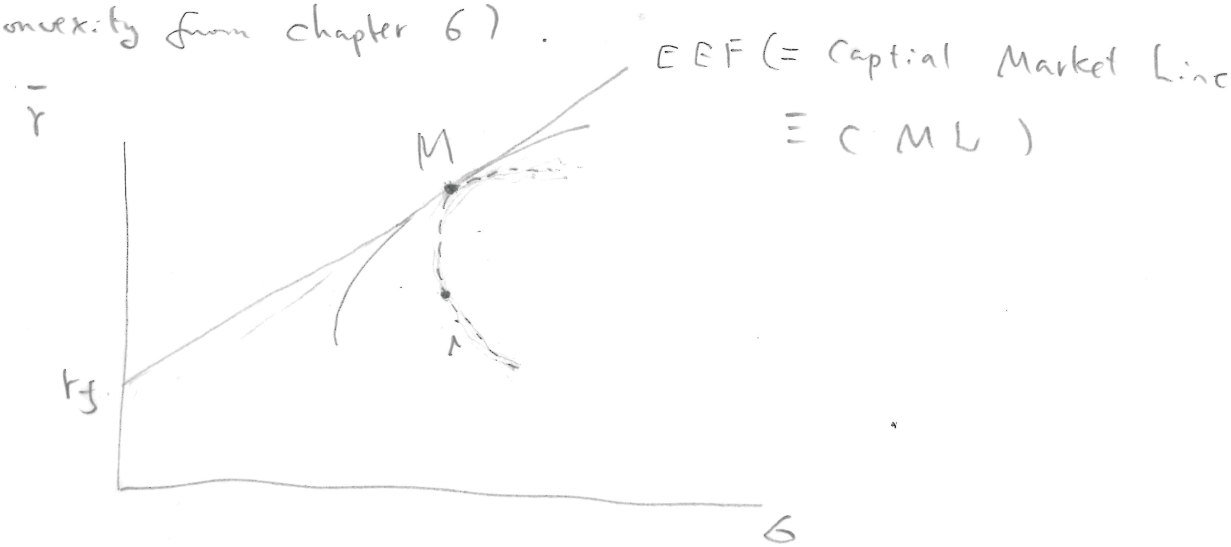
For a 2-asset portfolio that invests α in asset i and $(1-\alpha)$ in M. The expected portfolio return is:

$$\bar{r}_\alpha = \alpha \bar{r}_i + (1-\alpha) \bar{r}_M \qquad \bar{r} \equiv E(r).$$

The std. of the portfolio is:

$$\sigma_\alpha = [\alpha^2 \sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2 \sigma_M^2]^{1/2}$$

As α varies, we have the bullet-shape feasible set ($\because$ convexity from chapter 6).



The $\widehat{iM}$ curve must be tangent to the EEF.

i.e. the slope of $\widehat{iM}$ @ M = the slope of the EEF

$$\left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{\bar{r}_M - r_f}{\sigma_M} \quad \cdots\cdots (1)$$

To figure out the slope of $\widehat{iM}$ @ M, first we recognize the following fact: @ M, $\alpha = 0$; that is, we put 100% in M.

A little bit of Calculus :

$1 - 2\alpha + \alpha^2$

$$\frac{d \bar{r}_a}{d\alpha} = \frac{d}{d\alpha}\left(\alpha \bar{r}_i + (1-\alpha)\bar{r}_M\right)$$

$2\alpha - 2\alpha^2$

$$= \bar{r}_i - \bar{r}_M \quad \cdots\cdots\cdots \quad (2)$$

Now, we write $\sigma_\alpha = [\cdot]^{1/2}$ where $\cdot \equiv \alpha^2 \sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2 \sigma_M^2$

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{1}{2}(\cdot)^{-\frac{1}{2}} \cdot \left(2\alpha\sigma_i^2 + (2\sigma_{iM} - 4\alpha\sigma_{iM}) + (-2+2\alpha)\sigma_M^2\right)$$

$f(x) = h(g(x))$

$f'(x) = h'(g(x)) \cdot g'(x)$

Chain Rule

$$= \frac{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)\sigma_M^2}{(\cdot)^{1/2}}$$

$$= \frac{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)\sigma_M^2}{\sigma_\alpha}$$

@ M , i.e. $\alpha = 0$

$$\left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} = \frac{0 + (1-0)\sigma_{iM} + (-1)\sigma_M^2}{\sigma_\alpha}$$

$= \sigma_M$ @ M

$$= \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M} \quad \cdots\cdots\cdots \quad (3)$$

The slope of iM @ M (when $\alpha = 0$) is:

$$\left.\frac{d\,\bar{r}_\alpha}{d\,\sigma_\alpha}\right|_{\alpha=0} = \left.\frac{\overset{(2)}{d\bar{r}_\alpha / d\alpha}}{\underset{(3)}{d\sigma_\alpha / d\alpha}}\right|_{\alpha=0} \qquad \text{i.e. the Chain rule}$$

$$= \frac{\bar{r}_i - \bar{r}_M}{\left(\dfrac{\sigma_{iM} - \sigma_M^2}{\sigma_M}\right)} = \frac{(\bar{r}_i - \bar{r}_M)\,\sigma_M}{\sigma_{iM} - \sigma_M^2}$$

Now, Use (1):

$$\text{The slope of } iM \text{ @ } M = \frac{\bar{r}_M - r_f}{\sigma_M}$$

$$\therefore \frac{(\bar{r}_i - \bar{r}_M)\,\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\bar{r}_M - r_f}{\sigma_M}$$

$$(\bar{r}_i - \bar{r}_M)\,\sigma_M^2 = (\bar{r}_M - r_f)(\sigma_{iM} - \sigma_M^2)$$

$$\bar{r}_i - \bar{r}_M = \frac{\bar{r}_M - r_f}{\sigma_M^2}(\sigma_{iM} - \sigma_M^2) \qquad \left[\beta \equiv \frac{\sigma_{iM}}{\sigma_M^2}\right]$$

$$\bar{r}_i = \frac{\bar{r}_M - r_f}{\sigma_M^2}\,\sigma_{iM} - \frac{\bar{r}_M - r_f}{\cancel{\sigma_M^2}}\,\cancel{\sigma_M^2} + \bar{r}_M$$

$$= \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - r_f) + r_f = r_f + \beta(\bar{r}_M - r_f) \quad \cdot \text{X}$$